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Fixed point theorems and ergodic theorems for nonlinear mappings in Banach spaces

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Abstract. In this paper, we first introduce a class of nonlinear mappings called generalized nonspreading which contains the class of nonspreading mappings in a Banach space and then prove a fixed point theorem, a nonlinear mean convergence theorem of Baillon's type and a weak convergence theorem of Mann's type for such nonlinear mappings in a Banach space. Using these theorems, we obtain some fixed point theorems, nonlinear mean convergence theorems and weak convergence theorems in a Banach space.

Key words: Banach space, fixed point, mean convergence, weak convergence, nonexpansive mapping, nonspreading mapping

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ and let C be a nonempty subset of H. Let T be a mapping of C into H. Then we denote by $F(T)$ the set of fixed points of T. A mapping $T : C \to H$ is

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called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. An important example of nonexpansive mappings in a Hilbert space is a firmly nonexpansive mapping. Let C be a nonempty subset of H. A mapping $F : C \to H$ is said to be firmly nonexpansive if

$$\|Fx - Fy\|^2 \le \langle x - y, Fx - Fy \rangle$$

for all $x, y \in C$; see, for instance, Browder [6] and Goebel and Kirk [8]. It is known that a firmly nonexpansive mapping F can be deduced from an equilibrium problem in a Hilbert space; see, for instance, [4] and [7]. Recently, Kohsaka and Takahashi [26], and Takahashi [35] introduced the following nonlinear mappings which are deduced from a firmly nonexpansive mapping in a Hilbert space. A mapping $T : C \to H$ is called nonspreading [26] if

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. A mapping $T : C \to H$ is called hybrid [35] if

$$3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. They proved fixed point theorems for such mappings; see also Kohsaka and Takahashi [25] and Iemoto and Takahashi [17]. Recently, Kocourek, Takahashi and Yao [22] defined a broad class of nonlinear mappings containing the classes of nonexpansive mappings, nonspreading mappings and hybrid mappings in a Hilbert space: A mapping $T : C \to H$ is called generalized hybrid [22] if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{1.1}$$

for all $x, y \in C$. We call such a mapping an (α, β)-generalized hybrid mapping. Then, Kocourek, Takahashi and Yao [22] proved a fixed point theorem for such mappings in a Hilbert space. Further, they proved a nonlinear mean convergence theorem of Baillon's type [3] in a Hilbert space.

In this paper, motivated by these results, we first introduce a class of nonlinear mappings called generalized nonspreading which contains the class of nonspreading mappings in a Banach space and then prove a fixed point theorem, a nonlinear mean convergence theorem of Baillon's type and a weak convergence theorem of Mann's type for such nonlinear mappings in a Banach space. Using these theorems, we obtain some fixed point theorems, nonlinear mean convergence theorems and weak convergence theorems in a Banach space.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the topological dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon\right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let C be a nonempty subset of a Banach space E. A mapping $T : C \to E$ is nonexpansive [5, 9, 19] if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to E$ is quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - y\| \le \|x - y\|$ for all $x \in C$ and $y \in F(T)$, where $F(T)$ is the set of fixed points of T. If C is a nonempty closed convex subset of a strictly convex Banach space E and $T : C \to C$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [18]. Let E be a Banach space. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit (2.1) is attained uniformly for $x \in U$. It is also said to be Fréchet differentiable if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. A Banach space E is called uniformly smooth if the limit (2.1) is attained uniformly for $x, y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then J is uniformly norm-to-weak* continuous on each bounded subset of E, and if the norm of E is Fréchet differentiable, then J is norm-to-norm continuous. If E is uniformly smooth, J is uniformly norm-to-norm continuous on each bounded subset of E. For more details, see [31, 32]. The following result is also well known; see [32].

Theorem 2.1. *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x - y, Jx - Jy\rangle \geq 0$ for all $x, y \in E$. Further, if E is strictly convex and $\langle x - y, Jx - Jy\rangle = 0$, then $x = y$.*

Let E be a smooth Banach space. The function $\phi\colon E \times E \to (-\infty, \infty)$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for $x, y \in E$, where J is the duality mapping of E; see [1] and [20]. We have from the definition of ϕ that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle \tag{2.2}$$

for all $x, y, z \in E$. From $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$, we can see that $\phi(x, y) \geq 0$. Further, we can obtain the following equality:

$$2\langle x - y, Jz - Jw\rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w) \tag{2.3}$$

for $x, y, z, w \in E$. If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \Longleftrightarrow x = y. \tag{2.4}$$

The following theorems are in Xu [39] and Kamimura and Takahashi [20].

Theorem 2.2 (Xu [39]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

for all $x, y \in B_r$ and λ with $0 \leq \lambda \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Theorem 2.3 (Kamimura and Takahashi [20]). *Let E be smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \leq \phi(x, y)$$

for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Let E be a smooth Banach space and let C be a nonempty subset of E. Then a mapping $T : C \to E$ is called generalized nonexpansive [11, 13, 15] if $F(T) \neq \emptyset$ and

$$\phi(Tx, y) \leq \phi(x, y)$$

for all $x \in C$ and $y \in F(T)$. Let D be a nonempty subset of a Banach space E. A mapping $R : E \to D$ is said to be sunny if

$$R(Rx + t(x - Rx)) = Rx$$

for all $x \in E$ and $t \geq 0$. A mapping $R : E \to D$ is said to be a retraction or a projection if $Rx = x$ for all $x \in D$. A nonempty subset D of a smooth Banach space E is said to be a generalized nonexpansive retract (resp. sunny generalized nonexpansive retract) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto D; see [10–12] for more details. The following results are in Ibaraki and Takahashi [11].

Theorem 2.4 (Ibaraki and Takahashi [11]). *Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Theorem 2.5 (Ibaraki and Takahashi [11]). *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x, z) \in E \times C$. Then the following hold:*

(i) $z = Rx$ *if and only if* $\langle x - z, Jy - Jz\rangle \leq 0$ *for all* $y \in C$;
(ii) $\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.

In 2007, Kohsaka and Takahashi [24] proved the following results:

Theorem 2.6 (Kohsaka and Takahashi [24]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E. Then the following are equivalent:*

(a) *C is a sunny generalized nonexpansive retract of E;*
(b) *C is a generalized nonexpansive retract of E;*
(c) *JC is closed and convex.*

Theorem 2.7 (Kohsaka and Takahashi [24]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed sunny generalized nonexpansive retract of E. Let R be the sunny generalized nonexpansive retraction from E onto C and let $(x, z) \in E \times C$. Then the following are equivalent:*

(i) $z = Rx$;
(ii) $\phi(x, z) = min_{y \in C} \phi(x, y)$.

Very recently, Ibaraki and Takahashi [16] also obtained the following result concerning the set of fixed points of a generalized nonexpansive mapping.

Theorem 2.8 (Ibaraki and Takahashi [16]). *Let E be a reflexive, strictly convex and smooth Banach space and let T be a generalized nonexpansive mapping from E into itself. Then, $F(T)$ is closed and $JF(T)$ is closed and convex.*

The following is a direct consequence of Theorems 2.6 and 2.8.

Theorem 2.9 (Ibaraki and Takahashi [16]). *Let E be a reflexive, strictly convex and smooth Banach space and let T be a generalized nonexpansive mapping from E into itself. Then, $F(T)$ is a sunny generalized nonexpansive retract of E.*

3. Fixed point theorems

In this section, we try to extend Kocourek, Takahashi and Yao's fixed point theorem [22] in a Hilbert space to that in a Banach space. Let E be a smooth Banach space, let C be a nonempty closed convex subset of E and let J be the duality mapping from E into E^*. Then, a mapping $T : C \to E$ is called generalized nonspreading if there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha\phi(Tx, Ty) + (1-\alpha)\phi(x, Ty) + \gamma\{\phi(Ty, Tx) - \phi(Ty, x)\} \\ &\leq \beta\phi(Tx, y) + (1-\beta)\phi(x, y) + \delta\{\phi(y, Tx) - \phi(y, x)\} \end{aligned} \tag{3.1}$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for $x, y \in E$. We call such a mapping an $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping. Let T be an $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping. Observe that if $F(T) \neq \emptyset$, then $\phi(u, Ty) \leq \phi(u, y)$ for all $u \in F(T)$ and $y \in C$. Indeed, putting $x = u \in F(T)$ in (3.1), we obtain

$$\phi(u, Ty) + \gamma\{\phi(Ty, u) - \phi(Ty, u)\} \leq \phi(u, y) + \delta\{\phi(y, u) - \phi(y, u)\}.$$

So, we have that

$$\phi(u, Ty) \leq \phi(u, y) \tag{3.2}$$

for all $u \in F(T)$ and $y \in C$. Further, if E is a Hilbert space, then we have $\phi(x, y) = \|x - y\|^2$ for $x, y \in E$. So, from (3.1) we obtain the following:

$$\begin{aligned} &\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 + \gamma\{\|Tx - Ty\|^2 - \|x - Ty\|^2\} \\ &\leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 + \delta\{\|Tx - y\|^2 - \|x - y\|^2\} \end{aligned} \tag{3.3}$$

for all $x, y \in C$. This implies that

$$
\begin{aligned}
&(\alpha+\gamma)\|Tx - Ty\|^2 + \{1-(\alpha+\gamma)\}\|x - Ty\|^2 \\
&\le (\beta+\delta)\|Tx - y\|^2 + \{1-(\beta+\delta)\}\|x - y\|^2
\end{aligned} \tag{3.4}
$$

for all $x, y \in C$. That is, T is a generalized hybrid mapping [22] in a Hilbert space. Now, using the technique developed by [30], we prove a fixed point theorem for generalized nonspreading mappings in a Banach space.

Theorem 3.1. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of E. Let T be a generalized nonspreading mapping of C into itself. Then, the following are equivalent:*

(a) $F(T) \neq \emptyset$;
(b) $\{T^n x\}$ *is bounded for some* $x \in C$.

Proof. Let T be a generalized nonspreading mapping of C into itselt. Then, there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$
\begin{aligned}
&\alpha\phi(Tx, Ty) + (1-\alpha)\phi(x, Ty) + \gamma\{\phi(Ty, Tx) - \phi(Ty, x)\} \\
&\le \beta\phi(Tx, y) + (1-\beta)\phi(x, y) + \delta\{\phi(y, Tx) - \phi(y, x)\}
\end{aligned}
$$

for all $x, y \in C$. If $F(T) \neq \emptyset$, then $\phi(u, Ty) \le \phi(u, y)$ for all $u \in F(T)$ and $y \in C$. So, if u is a fixed point in C, then we have $\phi(u, T^n x) \le \phi(u, x)$ for all $n \in \mathbb{N}$ and $x \in C$. This implies (a) $\Longrightarrow$ (b). Let us show (b) $\Longrightarrow$ (a). Suppose that there exists $x \in C$ such that $\{T^n x\}$ is bounded. Then for any $y \in C$ and $k \in \mathbb{N} \cup \{0\}$, we have

$$
\begin{aligned}
&\alpha\phi(T^{k+1}x, Ty) + (1-\alpha)\phi(T^k x, Ty) + \gamma\{\phi(Ty, T^{k+1}x) - \phi(Ty, T^k x)\} \\
&\le \beta\phi(T^{k+1}x, y) + (1-\beta)\phi(T^k x, y) + \delta\{\phi(y, T^{k+1}x) - \phi(y, T^k x)\} \\
&= \beta\{\phi(T^{k+1}x, Ty) + \phi(Ty, y) + 2\langle T^{k+1}x - Ty, JTy - Jy\rangle\} \\
&\quad + (1-\beta)\{\phi(T^k x, Ty) + \phi(Ty, y) + 2\langle T^k x - Ty, JTy - Jy\rangle\} \\
&\quad + \delta\{\phi(y, T^{k+1}x) - \phi(y, T^k x)\}.
\end{aligned} \tag{3.5}
$$

This implies that

$$
\begin{aligned}
0 &\le (\beta-\alpha)\{\phi(T^{k+1}x, Ty) - \phi(T^k x, Ty)\} + \phi(Ty, y) \\
&+ 2\langle \beta T^{k+1}x + (1-\beta)T^k x - Ty, JTy - Jy\rangle \\
&- \gamma\{\phi(Ty, T^{k+1}x) - \phi(Ty, T^k x)\} + \delta\{\phi(y, T^{k+1}x) - \phi(y, T^k x)\}.
\end{aligned} \tag{3.6}
$$

Summing up these inequalities (3.6) with respect to $k = 0, 1, \dots, n-1$, we have

$$
\begin{aligned}
&0 \le (\beta - \alpha)\{\phi(T^n x, Ty) - \phi(x, Ty)\} + n\phi(Ty, y) \\
&\quad + 2\langle x + Tx + \cdots + T^{n-1}x + \beta(T^n x - x) - nTy, JTy - Jy\rangle \\
&\quad - \gamma\{\phi(Ty, T^n x) - \phi(Ty, x)\} + \delta\{\phi(y, T^n x) - \phi(y, x)\}.
\end{aligned}
\tag{3.7}
$$

Dividing by n in (3.7), we have

$$
\begin{aligned}
0 \le\; &\frac{1}{n}(\beta - \alpha)\{\phi(T^n x, Ty) - \phi(x, Ty)\} + \phi(Ty, y) \\
&+ 2\langle S_n x + \frac{1}{n}\beta(T^n x - x) - Ty, JTy - Jy\rangle \\
&- \frac{1}{n}\gamma\{\phi(Ty, T^n x) - \phi(Ty, x)\} + \frac{1}{n}\delta\{\phi(y, T^n x) - \phi(y, x)\},
\end{aligned}
\tag{3.8}
$$

where $S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$. Since $\{T^n x\}$ is bounded by assumption, $\{S_n x\}$ is bounded. Thus we have a subsequence $\{S_{n_i} x\}$ of $\{S_n x\}$ such that $\{S_{n_i} x\}$ converges weakly to a point $u \in C$. Letting $n_i \to \infty$ in (3.8), we obtain

$$
0 \le \phi(Ty, y) + 2\langle u - Ty, JTy - Jy\rangle.
$$

Putting $y = u$, we obtain

$$
\begin{aligned}
0 &\le \phi(Tu, u) + 2\langle u - Tu, JTu - Ju\rangle \\
&= \phi(Tu, u) + \phi(u, u) + \phi(Tu, Tu) - \phi(u, Tu) - \phi(Tu, u) \\
&= -\phi(u, Tu).
\end{aligned}
\tag{3.9}
$$

Hence we have $\phi(u, Tu) \le 0$ and then $\phi(u, Tu) = 0$. Since E is strictly convex, we obtain $u = Tu$. Therefore $F(T)$ is nonempty. This completes the proof. □

Using Theorem 3.1, we have the following fixed point theorems in a Banach space.

Theorem 3.2 (Kohsaka and Takahashi [26]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a nonspreading mapping, i.e.,*

$$
\phi(Tx, Ty) + \phi(Ty, Tx) \le \phi(Tx, y) + \phi(Ty, x)
$$

for all $x, y \in C$*. Then, the following are equivalent:*

(a) $F(T) \neq \emptyset$;
(b) $\{T^n x\}$ *is bounded for some* $x \in C$.

Proof. Putting $\alpha = \beta = \gamma = 1$ and $\delta = 0$ in (3.1), we obtain that

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x)$$

for all $x, y \in C$. So, we have the desired result from Theorem 3.1. □

See [2, 21, 23, 33, 38] for examples and convergence theorems for nonspreading mappings in a Banach space.

Theorem 3.3. *Let* E *be a smooth, strictly convex and reflexive Banach space and let* C *be a nonempty closed convex subset of* E*. Let* $T : C \to C$ *be a hybrid mapping [35], i.e.,*

$$2\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x) + \phi(x, y)$$

for all $x, y \in C$*. Then, the following are equivalent:*

(a) $F(T) \neq \emptyset$;
(b) $\{T^n x\}$ *is bounded for some* $x \in C$.

Proof. Putting $\alpha = 1$, $\beta = \gamma = \frac{1}{2}$ and $\delta = 0$ in (3.1), we obtain that

$$2\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x) + \phi(x, y)$$

for all $x, y \in C$. So, we have the desired result from Theorem 3.1. □

Theorem 3.4. *Let* E *be a smooth, strictly convex and reflexive Banach space and let* C *be a nonempty closed convex subset of* E*. Let* $T : C \to C$ *be a mapping such that*

$$\alpha\phi(Tx, Ty) + (1 - \alpha)\phi(x, Ty) \leq \beta\phi(Tx, y) + (1 - \beta)\phi(x, y)$$

for all $x, y \in C$*. Then, the following are equivalent:*

(a) $F(T) \neq \emptyset$;
(b) $\{T^n x\}$ *is bounded for some* $x \in C$.

Proof. Putting $\gamma = \delta = 0$ in (3.1), we obtain that

$$\alpha\phi(Tx, Ty) + (1 - \alpha)\phi(x, Ty) \leq \beta\phi(Tx, y) + (1 - \beta)\phi(x, y)$$

for all $x, y \in C$. So, we have the desired result from Theorem 3.1. □

As a direct consequence of Theorem 3.4, we have Kocourek, Takahashi and Yao's fixed point theorem [22] in a Hilbert space.

Theorem 3.5 (Kocourek, Takahashi and Yao [22]). *Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \to C$ be a generalized hybrid mapping, i.e., there are $\alpha, \beta \in \mathbb{R}$ such that*

$$\alpha\|Tx-Ty\|^2+(1-\alpha)\|x-Ty\|^2 \leq \beta\|Tx-y\|^2+(1-\beta)\|x-y\|^2 \tag{3.10}$$

for all $x, y \in C$. Then, the following are equivalent:

(a) $F(T) \neq \emptyset$;
(b) $\{T^n x\}$ *is bounded for some* $x \in C$.

Proof. We know that $\phi(x, y) = \|x - y\|^2$ for all $x, y \in C$ in Theorem 3.4. So, we have the desired result from Theorem 3.4. □

4. Some properties of generalized nonspreading mappings

In this section, we first discuss the demiclosedness property of generalized nonspreading mappings in a Banach space. Let E be a Banach space and let C be a nonempty subset of E. Let $T : C \to C$ be a mapping. Then, $p \in C$ be an asymptotic fixed point of T [29] if there exists $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote by $\hat{F}(T)$ the set of asymptotic fixed points of T. A mapping T of C into itself is said to have the demiclosedness property on C if $\hat{F}(T) = F(T)$.

Proposition 4.1. *Let E be a strictly convex Banach space with a uniformly Gâteaux differentiable norm, let C be a nonempty closed convex subset of E and let T be a generalized nonspreading mapping of C into itself. Then $\hat{F}(T) = F(T)$.*

Proof. Since $T : C \to C$ is a generalized nonspreading mapping, there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha\phi(Tx, Ty) + (1-\alpha)\phi(x, Ty) + \gamma\{\phi(Ty, Tx) - \phi(Ty, x)\} \\ &\leq \beta\phi(Tx, y) + (1-\beta)\phi(x, y) + \delta\{\phi(y, Tx) - \phi(y, x)\} \end{aligned} \tag{4.1}$$

for all $x, y \in C$. The inclusion $F(T) \subset \hat{F}(T)$ is obvious. Thus we show $\hat{F}(T) \subset F(T)$. Let $u \in \hat{F}(T)$ be given. Then we have a sequence $\{x_n\}$ of C such that $x_n \rightharpoonup u$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Since the norm of E is uniformly Gâteaux differentiable, the duality mapping J on E is uniformly norm-to-weak* continuous on each bounded subset of E; see Takahashi [31]. Thus

$$\lim_{n\to\infty} \langle w, JTx_n - Jx_n\rangle = 0$$

for all $w \in E$. On the other hand, since $T : C \to C$ is a generalized nonspreading mapping, we obtain that

$$\begin{aligned}
&\alpha\phi(Tx_n, Tu) + (1-\alpha)\phi(x_n, Tu) + \gamma\{\phi(Tu, Tx_n) - \phi(Tu, x_n)\} \\
&\quad \le \beta\phi(Tx_n, u) + (1-\beta)\phi(x_n, u) + \delta\{\phi(u, Tx_n) - \phi(u, x_n)\} \\
&\quad = \beta\{\phi(Tx_n, Tu) + \phi(Tu, u) + 2\langle Tx_n - Tu, JTu - Ju\rangle\} \\
&\qquad + (1-\beta)\{\phi(x_n, Tu) + \phi(Tu, u) + 2\langle x_n - Tu, JTu - Ju\rangle\} \\
&\qquad + \delta\{\phi(u, Tx_n) - \phi(u, x_n)\}.
\end{aligned} \tag{4.2}$$

This implies that

$$\begin{aligned}
0 &\le (\beta-\alpha)\{\phi(Tx_n, Tu) - \phi(x_n, Tu)\} + \phi(Tu, u) \\
&\quad + 2\langle \beta Tx_n + (1-\beta)x_n - Tu, JTu - Ju\rangle \\
&\quad - \gamma\{\phi(Tu, Tx_n) - \phi(Tu, x_n)\} + \delta\{\phi(u, Tx_n) - \phi(u, x_n)\} \\
&= (\beta-\alpha)\{\|Tx_n\|^2 - \|x_n\|^2 - 2\langle Tx_n - x_n, JTu\rangle\} + \phi(Tu, u) \\
&\quad + 2\langle \beta(Tx_n - x_n) + x_n - Tu, JTu - Ju\rangle \\
&\quad - \gamma\{\|Tx_n\|^2 - \|x_n\|^2 - 2\langle Tu, JTx_n - Jx_n\rangle\} \\
&\quad + \delta\{\|Tx_n\|^2 - \|x_n\|^2 - 2\langle u, JTx_n - Jx_n\rangle\}.
\end{aligned} \tag{4.3}$$

From

$$\begin{aligned}
|\|Tx_n\|^2 - \|x_n\|^2| &= (\|Tx_n\| + \|x_n\|)|\|Tx_n\| - \|x_n\|| \\
&\le (\|Tx_n\| + \|x_n\|)\|Tx_n - x_n\|,
\end{aligned}$$

we have $\|Tx_n\|^2 - \|x_n\|^2 \to 0$ as $n \to \infty$. So, letting $n \to \infty$ in (4.3), we have that

$$\begin{aligned}
0 &\le \phi(Tu, u) + 2\langle u - Tu, JTu - Ju\rangle \\
&= \phi(Tu, u) + \phi(u, u) + \phi(Tu, Tu) - \phi(u, Tu) - \phi(Tu, u) \\
&= -\phi(u, Tu).
\end{aligned}$$

Thus $\phi(u, Tu) \le 0$ and then $\phi(u, Tu) = 0$. Since E is strictly convex, we obtain $u = Tu$. This completes the proof. □

From Matsushita and Takahashi [28], we also know the following result.

Lemma 4.2 (Matsushita and Takahashi [28]). *Let E be a smooth and strictly convex Banach space, let C be a nonempty closed convex subset of E and let T be a mapping of C into itself such that $F(T)$ is nonempty. Assume that*

$$\phi(u, Ty) \leq \phi(u, y)$$

for all $u \in F(T)$ and $y \in C$. Then $F(T)$ is closed and convex.

Using this lemma (Lemma 4.2) and (3.2), we have the following result.

Proposition 4.3. *Let E be a smooth and strictly convex Banach space, let C be a nonempty closed convex subset of E and let T be a generalized nonspreading mapping of C into itself such that $F(T)$ is nonempty. Then $F(T)$ is closed and convex.*

Proof. It is sufficient to consider the case that $F(T)$ is nonempty. Then we have from (3.2) that $\phi(u, Ty) \leq \phi(u, y)$ for all $u \in F(T)$ and $y \in C$. From Lemma 4.2, $F(T)$ is closed and convex. □

Let E be a reflexive, smooth and strictly convex Banach space. Let C be a nonempty subset of E. Matsushita and Takahashi [28] also gave the following definition: A mapping $T : C \to C$ is relatively nonexpansive if $F(T) \neq \emptyset$, $\hat{F}(T) = F(T)$ and

$$\phi(y, Tx) \leq \phi(y, x)$$

for all $x \in C$ and $y \in F(T)$. Using Proposition 4.1, we prove the following theorem.

Theorem 4.4. *Let E be a strictly convex Banach space with a uniformly Gâteaux differentiable norm, let C be a nonempty closed convex subset of E and let T be a generalized nonspreading mapping of C into itself such that $F(T)$ is nonempty. Then, T is relatively nonexpansive.*

Proof. By assumption, $F(T)$ is nonempty. Since T is a generalized nonspreading mapping of C into itself, we have that

$$\phi(y, Tx) \leq \phi(y, x)$$

for all $x \in C$ and $y \in F(T)$. From Proposition 4.1, we also have $\hat{F}(T) = F(T)$. Thus T is relatively nonexpansive. □

As a direct consequence of Theorem 4.4, we have the following result.

Theorem 4.5 (Kohsaka and Takahashi [26]). *Let E be a strictly convex Banach space with a uniformly Gâteaux differentiable norm, let C be a nonempty closed convex subset of E and let T be a nonspreading mapping of C into itself such that $F(T)$ is nonempty. Then, T is relatively nonexpansive.* □

Proof. An $(\alpha, \beta, \gamma, \delta)$-generalized hybrid mapping T of C into itself such that $\alpha = \beta = \gamma = 1$ and $\delta = 0$ is a nonspreading mapping. From Theorem 4.4, we have the desired result.

5. Nonlinear ergodic theorems

Let E be a smooth Banach space and let C be a nonempty closed convex subset of E. Let $T : C \to E$ be a generalized nonspreading mapping; see (3.1). This mapping has the property that if $u \in F(T)$ and $x \in C$, then $\phi(u, Tx) \leq \phi(u, x)$. This property can be revealed by putting $x = u \in F(T)$ in (3.1). Similarly, putting $y = u \in F(T)$ in (3.1), we obtain that for $x \in C$,

$$\begin{aligned} &\alpha\phi(Tx, u) + (1-\alpha)\phi(x, u) + \gamma\{\phi(u, Tx) - \phi(u, x)\} \\ &\leq \beta\phi(Tx, u) + (1-\beta)\phi(x, u) + \delta\{\phi(u, Tx) - \phi(u, x)\} \end{aligned} \tag{5.1}$$

and hence

$$(\alpha - \beta)\{\phi(Tx, u) - \phi(x, u)\} + (\gamma - \delta)\{\phi(u, Tx) - \phi(u, x)\} \leq 0. \tag{5.2}$$

Therefore, we have that $\alpha > \beta$ together with $\gamma \leq \delta$ implies that

$$\phi(Tx, u) \leq \phi(x, u).$$

Now, we can prove the following nonlinear ergodic theorem for generalized nonspreading mappings in a Banach space.

Theorem 5.1. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty closed convex sunny generalized nonexpansive retract of E. Let $T : C \to C$ be a generalized nonspreading mapping with $F(T) \neq \emptyset$ such that $\phi(Tx, u) \leq \phi(x, u)$ for all $x \in C$ and $u \in F(T)$. Let R be the sunny generalized nonexpansive retraction of E onto $F(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to an element q of $F(T)$, where $q = \lim_{n\to\infty} RT^n x$.

Proof. We know that since C is a sunny generalized nonexpansive retract of E, there exists the sunny generalized nonexpansive retraction P of E onto C. On the other hand, by assumption, $T : C \to C$ is a generalized nonexpansive mapping, i.e., $F(T) \neq \emptyset$ and

$$\phi(Tx, u) \le \phi(x, u)$$

for all $x \in C$ and $u \in F(T)$. Putting $S = TP$, we have that S is a generalized nonexpansive mapping of E into itself such that $F(S) = F(T)$. Indeed, it is obvious that $F(S) = F(T)$. We also have that for any $x \in E$ and $u \in F(S) = F(T)$,

$$\phi(Sx, u) = \phi(TPx, u) \le \phi(Px, u) \le \phi(x, u).$$

So, S is a generalized nonexpansive mapping of E into itself such that $F(S) = F(T)$. From Theorems 2.9 and 2.4, there exists the sunny generalized nonexpansive retraction R of E onto $F(T)$. From Theorem 2.7, this retraction R is characterized by

$$Rx = \arg\min_{u \in F(T)} \phi(x, u).$$

We also know from Theorem 2.5 that

$$0 \le \langle v - Rv, JRv - Ju \rangle, \quad \forall u \in F(T),\ v \in C.$$

Adding up $\phi(Rv, u)$ to both sides of this inequality, we have

$$\begin{aligned} \phi(Rv, u) &\le \phi(Rv, u) + 2\langle v - Rv, JRv - Ju \rangle \\ &= \phi(Rv, u) + \phi(v, u) + \phi(Rv, Rv) \\ &\quad - \phi(v, Rv) - \phi(Rv, u) \\ &= \phi(v, u) - \phi(v, Rv). \end{aligned} \tag{5.3}$$

Since $\phi(Tz, u) \le \phi(z, u)$ for any $u \in F(T)$ and $z \in C$, it follows that

$$\begin{aligned} \phi(T^n x, RT^n x) &\le \phi(T^n x, RT^{n-1} x) \\ &\le \phi(T^{n-1} x, RT^{n-1} x). \end{aligned}$$

Hence the sequence $\phi(T^n x, RT^n x)$ is nonincreasing. Putting $u = RT^n x$ and $v = T^m x$ with $n \le m$ in (5.3), we have from Theorem 2.3 that

$$\begin{aligned} g(\|RT^m x - RT^n x\|) &\le \phi(RT^m x, RT^n x) \\ &\le \phi(T^m x, RT^n x) - \phi(T^m x, RT^m x) \\ &\le \phi(T^n x, RT^n x) - \phi(T^m x, RT^m x), \end{aligned}$$

where g is a strictly increasing, continuous and convex real-valued function with $g(0) = 0$. From the properties of g, $\{RT^n x\}$ is a Cauchy sequence.

Therefore $\{RT^n x\}$ converges strongly to a point $q \in F(T)$ since $F(T)$ is closed from Theorem 2.8. Next, consider a fixed $x \in C$ and an arbitrary subsequence $\{S_{n_i} x\}$ of $\{S_n x\}$ convergent weakly to a point v. From the proof of the fixed point theorem (Theorem 3.1) we know that $v \in F(T)$. Rewriting the characterization of the retraction R, we have that

$$0 \leq \left\langle T^k x - RT^k x, JRT^k x - Ju \right\rangle$$

and hence

$$\begin{aligned}\left\langle T^k x - RT^k x, Ju - Jq \right\rangle &\leq \left\langle T^k x - RT^k x, JRT^k x - Jq \right\rangle \\ &\leq \|T^k x - RT^k x\| \cdot \|JRT^k x - Jq\| \\ &\leq K\|JRT^k x - Jq\|,\end{aligned}$$

where K is an upper bound for $\|T^k x - RT^k x\|$. Summing up these inequalities for $k = 0, 1, \ldots, n-1$, we arrive to

$$\left\langle S_n x - \frac{1}{n}\sum_{k=0}^{n-1} RT^k x, Ju - Jq \right\rangle \leq K\frac{1}{n}\sum_{k=0}^{n-1} \|JRT^k x - Jq\|,$$

where $S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$. Letting $n_i \to \infty$ and remembering that J is continuous, we get

$$\langle v - q, Ju - Jq \rangle \leq 0.$$

This holds for any $u \in F(T)$. Therefore $Rv = q$. But because $v \in F(T)$, we have $v = q$. Thus the sequence $\{S_n x\}$ converges weakly to the point q. □

Using Theorem 5.1, we obtain the following theorems.

Theorem 5.2. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm. Let $T : E \to E$ be an $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping such that $\alpha > \beta$ and $\gamma \leq \delta$. Assume that $F(T) \neq \emptyset$ and let R be the sunny generalized nonexpansive retraction of E onto $F(T)$. Then, for any $x \in E$,*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to an element q of $F(T)$, where $q = \lim_{n\to\infty} RT^n x$.

Proof. Since the identity mapping I is a sunny generalized nonexpansive retraction of E onto E, E is a nonempty closed, convex sunny generalized nonexpansive retract of E. We also know that $\alpha > \beta$ together with $\gamma \leq \delta$ implies that

$$\phi(Tx, u) \le \phi(x, u)$$

for all $x \in E$ and $u \in F(T)$. So, we have the desired result from Theorem 5.1. □

Theorem 5.3 (Kocourek, Takahashi and Yao [22]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a generalized hybrid mapping with $F(T) \neq \emptyset$ and let P be the metric projection of H onto $F(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element p of $F(T)$, where $p = \lim_{n\to\infty} PT^n x$.

Proof. Since C is a nonempty closed convex subset of H, there exists the metric projection of H onto C. In a Hilbert space, the metric projection of H onto C is equivalent to the sunny generalized nonexpansive retraction of E onto C. On the other hand, a generalized hybrid mapping $T : C \to C$ with $F(T) \neq \emptyset$ is quasi-nonexpansive, i.e.,

$$\phi(Tx, u) = \|Tx - u\|^2 \le \|x - u\|^2 = \phi(x, u)$$

for all $x \in C$ and $u \in F(T)$. So, we have the desired result from Theorem 5.1. □

Remark. We do not know whether a nonlinear ergodic theorem of Baillon's type for nonspreading mappings holds or not.

6. Weak convergence theorems

In this section, we prove a weak convergence theorem of Mann's iteration [34] for generalized nonspreading mappings in a Banach space. For proving it, we need the following lemma obtained by Takahashi and Yao [37].

Lemma 6.1 (Takahashi and Yao [37]). *Let E be a smooth and uniformly convex Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let $T : C \to C$ be a generalized nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n < 1$ and let $\{x_n\}$ be a sequence in C generated by $x_1 = x \in C$ and*

$$x_{n+1} = R_C(\alpha_n x_n + (1 - \alpha_n)Tx_n), \quad \forall n \in \mathbb{N},$$

where R_C is a sunny generalized nonexpansive retraction of E onto C. Then $\{R_{F(T)}x_n\}$ converges strongly to an element z of $F(T)$, where $R_{F(T)}$ is a sunny generalized nonexpansive retraction of C onto $F(T)$.

Using Lemma 6.1 and the technique developed by [14, 27], we prove the following theorem.

Theorem 6.2. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex sunny generalized nonexpansive retract of E. Let $T : C \to C$ be a generalized nonspreading mapping with $F(T) \neq \emptyset$ such that $\phi(Tx, u) \leq \phi(x, u)$ for all $x \in C$ and $u \in F(T)$. Let R be the sunny generalized nonexpansive retraction of E onto $F(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Then, a sequence $\{x_n\}$ generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N}$$

converges weakly to $z \in F(T)$, where $z = \lim_{n\to\infty} Rx_n$.

Proof. Let $m \in F(T)$. By the assumption, we know that T is a generalized nonexpansive mapping of C into itself. So, we have

$$\begin{aligned}\phi(x_{n+1}, m) &= \phi(\alpha_n x_n + (1 - \alpha_n)Tx_n, m)\\ &\leq \alpha_n\phi(x_n, m) + (1 - \alpha_n)\phi(Tx_n, m)\\ &\leq \alpha_n\phi(x_n, m) + (1 - \alpha_n)\phi(x_n, m)\\ &= \phi(x_n, m).\end{aligned}$$

So, $\lim_{n\to\infty} \phi(x_n, m)$ exists. So, we have that the sequence $\{x_n\}$ is bounded. This implies that $\{Tx_n\}$ is bounded. Put $r = \sup_{n\in\mathbb{N}}\{\|x_n\|, \|Tx_n\|\}$. Using Lemma 2.2, we have that

$$\begin{aligned}\phi(x_{n+1}, m) &= \phi(\alpha_n x_n + (1-\alpha_n)Tx_n, m)\\ &\leq \|\alpha_n x_n + (1-\alpha_n)Tx_n\|^2 - 2\langle\alpha_n x_n + (1-\alpha_n)Tx_n, Jm\rangle + \|m\|^2\\ &\leq \alpha_n\|x_n\|^2 + (1 - \alpha_n)\|Tx_n\|^2 - \alpha_n(1 - \alpha_n)g(\|Tx_n - x_n\|)\\ &\quad - 2\alpha_n\langle x_n, Jm\rangle - 2(1 - \alpha_n)\langle Tx_n, Jm\rangle + \|m\|^2\\ &= \alpha_n(\|x_n\|^2 - 2\langle x_n, Jm\rangle + \|m\|^2)\\ &\quad + (1 - \alpha_n)(\|Tx_n\|^2 - 2\langle Tx_n, Jm\rangle + \|m\|^2)\\ &\quad - \alpha_n(1 - \alpha_n)g(\|Tx_n - x_n\|)\\ &= \alpha_n\phi(x_n, m) + (1-\alpha_n)\phi(Tx_n, m) - \alpha_n(1-\alpha_n)g(\|Tx_n - x_n\|)\\ &\leq \alpha_n\phi(x_n, m) + (1 - \alpha_n)\phi(x_n, m) - \alpha_n(1 - \alpha_n)g(\|Tx_n - x_n\|)\\ &= \phi(x_n, m) - \alpha_n(1 - \alpha_n)g(\|Tx_n - x_n\|).\end{aligned}$$

Then, we obtain

$$\alpha_n(1-\alpha_n)g(\|Tx_n - x_n\|) \le \phi(x_n, m) - \phi(x_{n+1}, m).$$

From the assumption of $\{\alpha_n\}$, we have

$$\lim_{n\to\infty} g(\|Tx_n - x_n\|) = 0.$$

Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$ for some $v \in C$. Since E is uniformly convex and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, we have from Proposition 4.1 that v is a fixed point of T. Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup u$ and $x_{n_j} \rightharpoonup v$. We know that $u, v \in F(T)$. Put $a = \lim_{n\to\infty}(\phi(x_n, u) - \phi(x_n, v))$. Since

$$\phi(x_n, u) - \phi(x_n, v) = 2\langle x_n, Jv - Ju\rangle + \|u\|^2 - \|v\|^2,$$

we have $a = 2\langle u, Jv - Ju\rangle + \|u\|^2 - \|v\|^2$ and $a = 2\langle v, Jv - Ju\rangle + \|u\|^2 - \|v\|^2$. From these equalities, we obtain

$$\langle u - v, Ju - Jv\rangle = 0.$$

Since E is strictly convex, it follows that $u = v$; see [32]. Therefore, $\{x_n\}$ converges weakly to an element u of $F(T)$. On the other hand, we know from Lemma 6.1 that $\{R_{F(T)}x_n\}$ converges strongly to an element z of $F(T)$. From Lemma 2.5, we also have

$$\langle x_n - R_{F(T)}x_n, JR_{F(T)}x_n - Ju\rangle \ge 0.$$

So, we have $\langle u - z, Jz - Ju\rangle \ge 0$. Since J is monotone, we also have $\langle u - z, Jz - Ju\rangle \le 0$. So, we have $\langle u - z, Jz - Ju\rangle = 0$. Since E is strictly convex, we have $z = u$. This completes the proof. □

Using Theorem 6.2, we can prove the following weak convergence theorems.

Theorem 6.3. *Let E be a uniformly convex and uniformly smooth Banach space. Let $T : E \to E$ be an $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping such that $\alpha > \beta$ and $\gamma \le \delta$. Assume that $F(T) \ne \emptyset$ and let R be the sunny generalized nonexpansive retraction of E onto $F(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n < 1$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Then, a sequence $\{x_n\}$ generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad \forall n \in \mathbb{N}$$

converges weakly to $z \in F(T)$, where $z = \lim_{n\to\infty} Rx_n$.

Proof. Since the identity mapping I is a sunny generalized nonexpansive retraction of E onto E, E is a nonempty closed, convex sunny generalized nonexpansive retract of E. We also know that $\alpha > \beta$ together with $\gamma \leq \delta$ implies that

$$\phi(Tx, u) \leq \phi(x, u)$$

for all $x \in E$ and $u \in F(T)$. So, we have the desired result from Theorem 6.2. □

Theorem 6.4 (Kocourek, Takahashi and Yao [22]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a generalized hybrid mapping with $F(T) \neq \emptyset$ and let P be the metric projection of H onto $F(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Then, a sequence $\{x_n\}$ generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N}$$

converges weakly to $z \in F(T)$, where $z = \lim_{n\to\infty} Px_n$.

Proof. Since C is a nonempty closed convex subset of H, there exists the metric projection of H onto C. In a Hilbert space, the metric projection of H onto C is equivalent to the sunny generalized nonexpansive retraction of E onto C. On the other hand, a generalized hybrid mapping $T : C \to C$ with $F(T) \neq \emptyset$ is quasi-nonexpansive, i.e.,

$$\phi(Tx, u) = \|Tx - u\|^2 \leq \|x - u\|^2 = \phi(x, u)$$

for all $x \in C$ and $u \in F(T)$. So, we have the desired result from Theorem 6.2. □

Remark. We do not know whether a weak convergence theorem of Mann's type for nonspreading mappings holds or not.

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